Hidden variable models for quantum theory cannot have any local part

Roger Colbeck and Renato Renner
Institute for Theoretical Physics
ETH Zurich, Switzerland
colbeck@phys.ethz.ch renner@phys.ethz.ch

It was shown by Bell that no local hidden variable model is compatible with quantum mechanics. If, instead, one permits the hidden variables to be entirely non-local, then any quantum mechanical predictions can be recovered. In this paper, we consider general hidden variable models which can have both local and non-local parts. We then show the existence of (experimentally verifiable) quantum correlations that are incompatible with any hidden variable model having a non-trivial local part, such as the model proposed by Leggett.

I. INTRODUCTION

Consider a source emitting two particles, which travel to two detectors, located far apart. The detectors are controlled by Alice and Bob. We denote Alice's choice of measurement by A, and similarly Bob's by B. The measurement devices generate the outcomes X and Y on Alice's and Bob's sides, respectively.

In a hidden variable model, one attempts to describe the outcomes of such measurements by assuming that there are hidden random variables, in the following denoted by U, V, and W, distributed according to some joint probability distribution P_{UVW}. (For reasons to be clarified below, we consider three different hidden variables.) Measurement outcomes then only depend on Alice and Bob's choice of measurement A and B as well as the values of the hidden variables U, V, W, that is, formally

$$X = f(A, B, U, V, W)$$
$$Y = g(A, B, U, V, W)$$

for some functions f and g. Rephrased in the language of conditional probability distributions, these conditions read

$$P_{X|A=a,B=b,U=u,V=v,W=w}(x) = \delta_{x,f(a,b,u,v,w)}$$
$$P_{Y|A=a,B=b,U=u,V=v,W=w}(y) = \delta_{y,g(a,b,u,v,w)} \ .$$

In this work, we divide the hidden variables into local and non-local parts:[1] U and V are, respectively, Alice's and Bob's *local* hidden variables, and W is a *non-local* hidden variable. The requirement is that, when the non-local part W is ignored, Alice's distribution depends only on the local parameters A, U and Bob's only on B, V,

$$\sum_w P_W(w) P_{X|A=a,B=b,U=u,V=v,W=w} \equiv P_{X|A=a,B=b,U=u,V=v} \equiv P_{X|A=a,U=u} \qquad (1)$$

$$\sum_w P_W(w) P_{Y|A=a,B=b,U=u,V=v,W=w} \equiv P_{Y|A=a,B=b,U=u,V=v} \equiv P_{Y|B=b,V=v}. \qquad (2)$$

We stress here that identities (1) and (2) do not restrict the generality of the model; they are merely a definition of what we call *local*. In fact, any possible dependence of the individual measurement outcomes X and Y on the choice of measurements A and B—in particular, the predictions of quantum theory—can be recreated by an appropriate choice of functions f and g that depend on the non-local variable W but not on the local variables U and V. In the following, we call such a model *entirely non-local*. The *de Broglie-Bohm theory* (see, e.g., [2]) is an example of such a model.

In the *Bell model* [2], one makes the assumption that the individual measurement outcomes are fully determined by local parameters, i.e., that the functions f and g only depend on the local variables U and V, respectively,

[1] Notice that our definition of *local* and *non-local* parts is not the same as that used in [1]. While ours is based on a distinction between local and non-local hidden variables, the definition in [1] relies on a convex decomposition of the conditional probability distribution into a local conditional distribution and a non-local one.

but not on W. It is well known that such an assumption is inconsistent with quantum theory. Modulo a few loopholes (see for example [3, 4, 5] for discussions), experiment agrees with the predictions of quantum mechanics, and hence falsifies Bell's model.

Leggett [6] has introduced a hidden variable model for which the hidden variables have both a local and a global part as above. In addition, he assumes that the expectation values of the measurement outcomes obey a specific law (Malus' law), which depends only on local quantities. More concretely, the assumption is that the measurement outcomes X and Y are binary values and that the measurement choices $A = \vec{A}$ and $B = \vec{B}$ as well as the local hidden variables $U = \vec{U}$ and $V = \vec{V}$ are unit vectors. The conditional probability distributions $P_{X|A=a,U=u}$ and $P_{Y|B=b,V=v}$ on the r.h.s. of (1) and (2) are given by $[|\vec{A}\cdot\vec{U}|^2, 1-|\vec{A}\cdot\vec{U}|^2]$ and $[|\vec{B}\cdot\vec{V}|^2, 1-|\vec{B}\cdot\vec{V}|^2]$, respectively. Such a model is inconsistent with quantum theory, and has motivated recent experiments [7, 8, 9].

In this paper, we show that there exist quantum correlations for which *all* hidden variable models with a non-trivial local part are inconsistent. More precisely, we show that there is a Bell-type experiment with binary outcomes[2] X and Y such that $P_{X|A=a,U=u} = \mathcal{U}_X$ for all a and u, and $P_{Y|B=b,V=v} = \mathcal{U}_Y$ for all b and v are the only distributions compatible with quantum mechanics, where $\mathcal{U}_X$ and $\mathcal{U}_Y$ denote the uniform distributions on X and Y, respectively. In particular, X and Y are independent of the local hidden variables U and V. Thus, the only hidden variable model compatible with quantum mechanical predictions is entirely non-local. This is in agreement with a similar result obtained independently by Branciard *et al.* [10], where it is shown that Leggett-type inequalities have no local part.

II. DEFINITIONS AND USEFUL LEMMAS

Our technical theorem will rely on the notion of non-signaling distributions. Intuitively, a conditional distribution $P_{XY|AB}$ is non-signaling if the behavior on Bob's side, specified by B and Y, cannot be influenced by Alice's choice of A, and vice versa. We give a general definition for n parties.

Definition 1. An n party conditional probability distribution $P_{X_1,\ldots,X_n|A_1,\ldots,A_n}$ is *non-signaling* if, for all subsets $S \subseteq \{1,\ldots,n\}$, we have

$$P_{X_{S_1},\ldots,X_{S_{|S|}}|A_1,\ldots,A_n} = P_{X_{S_1},\ldots,X_{S_{|S|}}|A_{S_1},\ldots,A_{S_{|S|}}} .$$

In the following, we denote by $D(P_X, Q_X)$ the *statistical distance* between two probability distributions P_X and Q_X, defined by $D(P_X, Q_X) = \frac{1}{2}\sum_x |P_X(x) - Q_X(x)|$. It is easy to verify that

$$D(P_X, Q_X) = \sum_x \max[0, Q_X(x) - P_X(x)] \ . \tag{3}$$

Furthermore, taking marginals cannot increase the statistical distance, i.e.,

$$D(P_X, Q_X) \leq D(P_{XZ}, Q_{XZ}) \ , \tag{4}$$

where P_X and Q_X are the marginals of joint distributions P_{XZ} and Q_{XZ}, respectively. In fact, if the marginals P_Z and Q_Z are equal, then the distance $D(P_{XZ}, Q_{XZ})$ can be written as the expectation of the distance between the corresponding conditional probability distributions,

$$D(P_{XZ}, Q_{XZ}) = \sum_z P_Z(z) D(P_{X|Z=z} Q_{X|Z=z}) \ . \tag{5}$$

Finally, we will use the following lemma which relates the statistical distance to the probability that two random variables take the same value.

Lemma 1. *Given a joint probability distribution P_{XY}, the distance between the marginals P_X and P_Y is upper bounded by the probability that $X \neq Y$, that is, $D(P_X, P_Y) \leq \sum_{x\neq y} P_{XY}(x, y)$.*

Proof. Define X' as a copy of X, so that $P_{XX'}(x, y) = 0$ for all $x \neq y$. Using (4) and (3), we have

$$D(P_X, P_Y) \leq D(P_{XX'}, P_{XY}) = \sum_{x\neq y} P_{XY}(x, y). \tag{6}$$

□

[2] Note that the labeling of the outcomes X and Y is irrelevant for the argument. For concreteness, one might think of $X \in \{-1, 1\}$ or $X \in \{0, 1\}$.

III. CHAINED BELL INEQUALITIES

We use the family of Bell inequalities introduced by Pearle [11] and Braunstein and Caves [12]. Each member of this family is indexed by $N \in \mathbb{N}$, the number of measurement choices. Alice can choose the measurements $A \in \{0, 2, \ldots, 2N-2\}$ and Bob $B \in \{1, 3, \ldots, 2N-1\}$. Each measurement has two outcomes, i.e., X and Y are binary. If x is one outcome, $\bar{x}$ denotes the other.

The quantity we consider is

$$I_N \equiv I_N(P_{XY|AB}) := \sum_{\substack{a,b \\ |a-b|=1}} \sum_x P_{XY|A=a,B=b}(x,\bar{x}) + \sum_x P_{XY|A=0,B=2N-1}(x,x) \ . \tag{7}$$

Note that, for any fixed a, b, the sum $\sum_x P_{XY|A=a,B=b}(x,\bar{x})$ corresponds to the probability that the values X and Y are distinct. It is easy to verify (but we are not going to use this fact) that all classical correlations satisfy $I_N \geq 1$ (i.e., $I_N \geq 1$ is a Bell inequality), and that the CHSH inequality [13] is the $N = 2$ version. We also emphasize that the bound $I_N \geq 1$ is independent of the actual measurements chosen and hence allows a device-independent falsification of hidden variable models (in contrast to Leggett-type inequalities).

Using a quantum mechanical setup, one can obtain a value of I_N, denoted I_N^{QM}, which is arbitrarily small in the large N limit. To see this, suppose Alice and Bob share the state $1/\sqrt{2}\,(|00\rangle + |11\rangle)$, and their measurements take the form of projections onto the states $\cos\frac{\theta_i}{2}\,|0\rangle + \sin\frac{\theta_i}{2}\,|1\rangle$ and $\sin\frac{\theta_i}{2}\,|0\rangle - \cos\frac{\theta_i}{2}\,|1\rangle$, where $\theta_i = \frac{i\pi}{2N}$ (Alice's measurements take $i = a$, and Bob's take $i = b$). Using this setup, the probability that Alice and Bob's measurement outcomes X and Y are distinct, for $|a - b| = 1$, is given by

$$\sum_x P_{XY|A=a,B=b}(x,\bar{x}) = \sin^2\frac{\pi}{4N}$$

and, likewise, the probability that the outcomes are equal for $a = 0$ and $b = 2N-1$ is

$$\sum_x P_{XY|A=0,B=2N-1}(x,x) = \sin^2\frac{\pi}{4N} \ .$$

Thus, quantum mechanics predicts

$$I_N^{\mathrm{QM}} = 2N\sin^2\frac{\pi}{4N} \ , \tag{8}$$

which, in the limit of large N, is approximated by $\frac{\pi^2}{8N}$ and can be made arbitrarily small.

IV. TECHNICAL RESULT

Our argument is based on a straightforward extension of a result about non-signaling distributions $P_{XY|AB}$ by Barrett, Kent, and Pironio [1]. The main difference between their result and our Theorem 1 is that our statement holds with respect to an additional third party with an input C and output Z. (When applying the theorem, the local hidden variables will take the place of Z, whereas C is not used.)

For the following, let X and Y be binary, $A \in \{0, 2, \ldots, 2N-2\}$, $B \in \{1, 3, \ldots, 2N-1\}$ for some $N \in \mathbb{N}$, as in Section III, and let Z and C be arbitrary.

Theorem 1. *If $P_{XYZ|ABC}$ is non-signaling then, for any C chosen independently of the inputs A and B,*[3]

$$D(P_{XZC|A=a}, \mathcal{U}_X \times P_{ZC}) \leq \frac{I_N}{2} \qquad \text{and} \qquad D(P_{YZC|B=b}, \mathcal{U}_Y \times P_{ZC}) \leq \frac{I_N}{2}$$

for all a, b, where $I_N \equiv I_N(P_{XY|AB})$, and where $\mathcal{U}_X$ and $\mathcal{U}_Y$ denote the uniform distributions on X and Y, respectively.

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[3] Because C is chosen independently of A and B, the joint distribution $P_{XYZC|AB}$ is given by $P_{XYZC|A=a,B=b}(x,y,z,c) = P_{XYZ|A=a,B=b,C=c}(x,y,z)P_C(c)$.

Proof. Using Lemma 1 and the triangle inequality, we have for any fixed z and c

$$\begin{aligned} I_N(P_{XY|AB,C=c,Z=z}) &= \sum_{\substack{a,b \\ |a-b|=1}} \sum_x P_{XY|A=a,B=b,C=c,Z=z}(x,\bar{x}) + \sum_x P_{XY|A=0,B=2N-1,C=c,Z=z}(x,x) \\ &\geq \sum_{\substack{a,b \\ |a-b|=1}} D(P_{X|A=a,C=c,Z=z}, P_{Y|B=b,C=c,Z=z}) + D(P_{\bar{X}|A=0,C=c,Z=z}, P_{Y|B=2N-1,C=c,Z=z}) \\ &\geq D(P_{\bar{X}|A=0,C=c,Z=z}, P_{X|A=0,C=c,Z=z}) \ . \end{aligned} \tag{9}$$

Then, since

$$D(P_{\bar{X}|A=0,C=c,Z=z}, P_{X|A=0,C=c,Z=z}) = 2D(P_{X|A=0,C=c,Z=z}, \mathcal{U}_X), \tag{10}$$

we obtain

$$D(P_{X|A=0,C=c,Z=z}, \mathcal{U}_X) \leq \frac{1}{2} I_N(P_{XY|AB,C=c,Z=z}) \ . \tag{11}$$

Taking the average over z and c (distributed according to $P_{ZC} \equiv P_{Z|C}P_C$) on both sides of this inequality and using (5) we conclude

$$D(P_{XZC|A=0}, \mathcal{U}_X \times P_{ZC}) \leq \frac{I_N}{2} \ .$$

The claim for arbitrary a (rather than $a = 0$) as well as the second inequality of the theorem follow by symmetry. □

For our argument, we apply the theorem to the setup described in Section III, with $Z := (U, V)$ and C equal to a constant (i.e., C is not used). Under the assumption that the hidden variables U and V are independent of the inputs a and b,[4] we have $P_{Z|A=a,B=b} \equiv P_Z$. This together with (1) and (2) implies the non-signaling condition. Theorem 1 thus gives

$$D(P_{XU|A=a}, \mathcal{U}_X \times P_U) \leq \frac{I_N}{2} \qquad \text{and} \qquad D(P_{YV|B=b}, \mathcal{U}_Y \times P_V) \leq \frac{I_N}{2} \tag{12}$$

for all a and b. In particular, for $I_N \ll 1$, the bound implies that the measurement outcomes X and Y are virtually independent of the local hidden variables U and V.

V. IMPLICATIONS

Before summarizing the implications of Theorem 1, we first stress that the contribution of this work is not a technical one. Our aim is to establish a connection between an argument proposed in [1] and recent work on hidden variable models, in particular Leggett-type models [6, 7, 8, 9, 10].

Suppose an experiment is performed, using the setup described in Section III, which allows us to estimate an upper bound I_N^* on the quantity $I_N \equiv I_N(P_{XY|AB})$ defined by (7). Then, according to (12), the *maximum locality* of X, which we measure in terms of its dependence on the local hidden variable U via $D(P_{XU|A=a}, \mathcal{U}_X \times P_U)$, is bounded by $I_N^*/2$.

For example, after many (noiseless) measurements of the CHSH quantity, I_4, one would eventually get an upper bound I_4^* close to $I_4^{\mathrm{QM}} = 2 - \sqrt{2}$ (see Eqn. (8)). Hence, the maximum locality of a hidden variable theory compatible with these measurements is $1 - 1/\sqrt{2} \approx 0.3$. This bound can be brought closer to zero by performing experiments according to the setup described in Section III with larger N.[5] Such experiments were proposed in [1].

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[4] This assumption simply says that, in an experiment, the choice of measurements a and b must not depend on the value of the local hidden variables. Of course, this is the case if the measurements are chosen at random.

[5] For any given practical setup, the optimal value of N which minimizes the upper bound I_N^* may depend on the specific noise model of the measurement devices.

In the limit of large N, quantum mechanics predicts $I_\infty^{\mathrm{QM}} = 0$. Hence, for any hidden variable model to describe these quantum correlations, we require $P_{XU|A=a} = \mathcal{U}_X \times P_U$, and $P_{YV|B=b} = \mathcal{U}_Y \times P_V$. Consequently, the outcomes X and Y for any fixed pair of measurements (a, b) are fully independent of the local hidden variables U and V. Notice that, we can reach this conclusion using only measurements in *one* plane of the Bloch sphere on each side (where Alice's plane contains a and Bob's b).

Finally, we discuss the implications for Leggett's model. Using our inequality (12) with N measurements in one plane of the Bloch sphere we conclude in the limit of large N that the model can only be consistent with the predictions of quantum mechanics if $\vec{U}$ and $\vec{V}$ are almost orthogonal to the measurement plane. Hence, with measurements in only one plane, we can establish that the local hidden variables $\vec{U}$ and $\vec{V}$ play no rôle. A further advantage of the inequality we use over those of the Leggett-type is that our inequalities enable a device independent falsification of any hidden variable model with non-trivial local part. Conversely, with the usual Leggett-type inequalities, the bound depends on the setup, and is hence less experimentally robust.

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